MAT 4575 Devoir #2 M. ALVO

Sur 17

1. #6.2.7 (Sur 6) a) $f(x;\theta) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}$ $x>0,\ \theta>0,\ \alpha>0$

$\ell(\theta) = \ln f(x;\theta) = (\alpha-1)\ln x - \frac{x}{\theta} - \ln\Gamma(\alpha) - \alpha\ln\theta$

$\ell'(\theta) = \frac{x}{\theta^2} - \frac{\alpha}{\theta} \Longrightarrow$ Pour un échantillon $\ell'(\theta) = 0 \Rightarrow \hat\theta = \frac{\sum x_i}{n\alpha}$

$\ell''(\theta) = -\frac{2x}{\theta^3} + \frac{\alpha}{\theta^2}$

$I(\theta) = -E\ell''(\theta) = \frac{2EX}{\theta^3} - \frac{\alpha}{\theta^2} = \frac{2\alpha\theta}{\theta^3} - \frac{\alpha}{\theta^2} = \frac{\alpha}{\theta^2}$ (p158)

b) $\operatorname{Var}\hat\theta = \frac{\operatorname{Var}X}{n\alpha^2} = \frac{\alpha\theta^2}{n\alpha^2} = \frac{\theta^2}{n\alpha}$ (p158)

Borne inférieur Cramer Rao $= \frac{1}{n\,I(\theta)} = \frac{\theta^2}{n\alpha} \Longrightarrow \hat\theta$ est efficace.

c) $\sqrt{n}(\hat\theta - \theta) \xrightarrow{D} N(0, I^{-1}(\theta)) = N\left(0, \frac{\theta^2}{\alpha}\right)$

2. #6.2.9 (Sur 4) $E(X) = \int_0^\infty x\frac{3\theta^3}{(x+\theta)^4}\,dx = \frac{\theta}{2}$ (intégration en partie)

$EY = 2E\bar X = 2EX = 2\cdot\frac{\theta}{2} = \theta \longrightarrow$ nonbiaisé

$\ell(\theta) = \log f(x;\theta) = 3\ln(3\theta) - 4\ln(x+\theta)$

$\ell'(\theta) = \frac{3}{\theta} - \frac{4}{x+\theta}$

$\ell''(\theta) = -\frac{3}{\theta^2} + \frac{4}{(x+\theta)^2}$

$I(\theta) = -E\ell''(\theta) = \frac{3}{\theta^2} - 4E\frac{1}{(X+\theta)^2} = \frac{3}{5\theta^2}$

Borne inférieure de Cramér Rao $= \dfrac{1}{n\, I(\theta)} = \dfrac{5\theta^2}{3n}$

Par contre $Var(2\bar{X}) = 4\, Var\, \bar{X} = \dfrac{4\, Var\, X}{n}$

$$E(X+\theta)^2 = 3\theta^3 \int_0^{\infty} \frac{(x+\theta)^2\, dx}{(x+\theta)^4} = 3\theta^3 \int_\theta^{\infty} \frac{1}{u^2}\, du = 3\theta^2$$

Mais $E(X+\theta)^2 = EX^2 + 2\theta EX + \theta^2$

$$= EX^2 + \theta^2 + \theta^2 = 3\theta^2 \Rightarrow EX^2 = 3\theta^2 - 2\theta^2$$

$$Var\, X = EX^2 - (EX)^2 = 3\theta^2 - 2\theta^2 - \left(\frac{\theta}{2}\right)^2 = \frac{3\theta^2}{4}$$

$$\Rightarrow Var(2\bar{X}) = 4 \cdot \frac{3\theta^2}{4n} = \frac{3\theta^2}{n}$$

$$\text{Efficacité de } (2\bar{X}) = \frac{\text{Information } (2\bar{X})}{\text{Information (Fisher)}} = \frac{n/3\theta^2}{3n/5\theta^2} = \frac{5}{3(3)} = \frac{5}{9}$$

3. # 6.4.3 $\quad L(\theta) = \prod f(x_i; \theta) \quad , \quad \theta = (\theta_1, \theta_2)$

(sur 3)

$$= \frac{1}{\theta_2^n}\, e^{-\sum (x_i - \theta_1)/\theta_2} \qquad \theta_1 \le \min x_i, \quad -\infty < \theta_2 < \infty$$

$$\Rightarrow \quad \hat{\theta}_1 = \min x_i$$

Aussi

$$\ell(\theta) = \ln L(\theta)$$

$$\frac{\partial \ell(\theta)}{\partial \theta_2} = -\frac{n}{\theta_2} + \frac{\sum (x_i - \theta_1)}{\theta_2^2} = 0$$

$$\Longrightarrow \quad \hat{\theta}_2 = \frac{\sum (x_i - \hat{\theta}_1)}{n}$$

4. #6.4.6 $\quad$ On sait que le max. de vraisemblance d'une fonction $U(\theta)$ est $U(\hat{\theta})$.

(sur 4)

a) $P(X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) = 0.90$

les estimés de μ, σ sont $\bar{x}$ et S. Puisque

$$\frac{b-\mu}{\sigma} = \Phi^{-1}(0.90) = 1.282$$

$$\hat{b} = \hat{\sigma}\,\Phi^{-1}(0.90) + \hat{\mu}$$

$$= S \cdot \Phi^{-1}(0.90) + \bar{x}$$

b). $P(X \le c) = \Phi\left(\frac{c-\mu}{\sigma}\right)$

$\Rightarrow$ L'estimé mle est $\Phi\left(\frac{c-\bar{x}}{S}\right)$.